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Interaction of light with the polarization devices: a vectorial Pauli algebraic approach

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Abstract

A theoretical approach to the interaction between polarized light and polarization devices, based on the vectorial and pure operatorial form of the Pauli algebra, is presented. Unlike the standard (Jones and Mueller) approaches, this formalism is coordinate-free, i.e. it does not appeal to any matrix representation of the involved operators. This vectorial approach establishes a mathematical bridge between the Hilbert space of the density operators of the polarization states and the Poincaré space of their geometric representations and gives a rigorous justification of the handling of the interactions between the polarization states and polarization systems on the Poincaré sphere (in the Poincaré ball). In such an approach, unlike the standard ones, the three relevant quantities that characterize the interaction—the gain, the Poincaré vector of the outgoing light and its degree of polarization—result straightforwardly, in block, in the Pauli vectorial expressions of the density operator of the output state. The final equations are symmetric, compact and physically expressive. A generalized form of Malus' law, for any dichroic device and partially polarized light is obtained this way.

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1. Introduction

The interaction of light with anisotropic media and polarization ('non-image-forming') optical devices has been studied theoretically for more than a century. The corresponding bibliography comprises thousands of papers and we cannot refer better to it than by means of some of the authoritative textbooks concerning the fundamentals of polarized light, e.g. [1–4]. From the large diversity of the up-to-date problems and approaches it is worth mentioning the group theoretical approach [5–7], the polarization dynamics in linear and nonlinear media [8–15], the analysis of the non-orthogonal optical devices [6–20] and that of the depolarizing media

[4, 16, 17], as reflecting some of the major trends in the field. Especially, the group theoretical approach has the great merit of revealing the fact that the linear polarization devices (and, more generally, anisotropic media) pertain to the very large class of 'two-state' or 'two-beam' systems [5, 6, 21–23], the algebra of all these systems being the same. This algebra is that of the linear operators defined on a unitary space of dimension two over the field of complex numbers $\mathbf{C}^1$.

In the polarization optics, the overwhelming majority of the papers makes use of the well-known matrix representations of this algebra: the Jones and the Mueller formalisms [1–7, 10, 13, 22, 23].

The matrix method has the pragmatic advantage of following a fixed, early learned and well-known routine, but the fundamental drawback of working with blind collections of numbers, associated more or less arbitrarily to the described reality. First, whereas an operator has, *per se*, a well-defined, a unique, mathematical identity, its matrix representation has a multitude of facets, corresponding to the multitude of bases in which it may be written.

In polarization optics, the pure operatorial ('coordinate-free' or 'matrix-free') approaches are somewhat isolated [20, 24–28].

Although paying tribute to the matrix language (even in the titles), some important papers handle the spectral, polar and the singular-value decompositions and their consequences in polarization optics in operatorial terms [29–32]. This fact has a very objective motivation: the observables (intensity, gain, degree of polarization) have invariant characters; they are independent of any coordinatization of the problem [33, 34]. 'Thinking in terms of scalar invariants in polarization optics has proved to be enormously fruitful in a number of physical contexts [33].'

It is well known that, besides the description of polarized light in the real physical space, there is another approach, in the abstract Hilbert space of the polarization states [35, 36]. The isomorphism of this space with the real unit ball Σ_3^1 underlies the famous, intuitive and effective Poincaré representation of the polarization states [1–4].

The Pauli algebra is the most widespread of the various mathematical tools (spherical trigonometry, quaternionic algebra, turns, Clifford algebra) of handling rotations in $\mathbf{R}^3$ and particularly on the Poincaré sphere. Therefore it was largely adopted in the polarization theory (e.g. [9–15, 26, 31, 32, 34, 37–40]), mainly in its scalar and matrix form (operating explicitly with the four Pauli matrices σ_i).

The aim of this paper is to give a vectorial and pure operatorial Pauli-algebraic treatment of the action of the deterministic 'canonical' [4] polarization devices (homogeneous phase shifters and homogeneous polarizers [30], [41]) on the arbitrarily polarized light. As we shall see, this approach provides the clearest mathematical justification of the intuitive geometric handling of the interactions states—systems on the Poincaré sphere: in this treatment both the density operators of the polarization states and of the polarization devices are characterized by some unit vectors—their Pauli axes [42]. To the Pauli axis of the density operator of the state corresponds, in the Poincaré sphere representation, the Poincaré vector (the three-dimensional, 'reduced' Stokes vector) of the state. Each action of a device operator on the density operator of a state (in the Hilbert space of the states) is mapped in the action of a corresponding operator which acts, in $\mathbf{R}^3$, on the Poincaré vector of the state, and moves its tip on the Poincaré sphere Σ_2^1 or in the Poincaré ball Σ_3^1.

A first advantage of this vectorial approach over the standard ones is that it leads straightforwardly, in the most direct manner and only in few lines of calculus to the whole group of three quantities which characterize the action of the system on the state: the gain g, the Poincaré unit vector $\mathbf{s}_\mathrm{o}$ of the polarization state and the degree of polarization p_o of the emergent light. For reaching them, the standard approaches make a long round about way:

they go through a more or less-explicit matrix representation of the operators and then come back to their invariants, the trace and the determinant, on the basis of which are expressed the gain and the degree of polarization [2–4]. Moreover, in the standard approaches, the gain, the output state and its degree of polarization are reached on separate lines of calculus. Here all the relevant quantities, $\mathbf{s}_o$, p_o, g arise in block in the expression of the polarization density operator of the output state. It is a unique expression which contains all information about the interaction which occurred.

We shall discuss in the conclusions section the roots of this compactness and simplicity.

For the sake of self-consistency of the paper, in section 2 we will deduce the Pauli-vectorial expressions of the operators of the various basic polarization devices: phase-shifters, partial and ideal polarizers.

In section 3, devoted to the birefringent devices, characterized by unitary operators, we shall show that to each unitary operator acting (in the Hilbert space of the states) on the polarization densities of the state, there corresponds an operator of $\mathbf{R}^3$ rotation acting in the Poincaré space on the corresponding Poincaré vector of the state.

Section 4 refers to the action of the dichroic devices on the partially polarized light. Particularly we obtain a very large generalization of Malus' law, for any dichroic device acting on partially polarized light, equation (53). The results obtained by this approach have a very simple and symmetric form—e.g. equations (53), (58) and (63). We shall discuss the reasons of this symmetry in conclusions.

2. Vectorial Pauli algebraic expansions of some operators widespread in polarization optics

2.1. Pauli algebraic expansion and the Pauli axis of a 2 × 2 operator

Let us consider a linear operator A defined on a unitary linear space of dimension two over the field of complex numbers $\mathbf{C}^1$. In the following we shall refer to these operators as 'two-dimensional operators', or '2 × 2 operators' as they are shortly called sometimes in the literature. Any such operator may be expanded in the basis of the Pauli σ_i operators in the form:

$$\mathsf{A} = \mathrm{a}_0\sigma_0 + \mathbf{a}\cdot\boldsymbol{\sigma}, \qquad (1)$$

where σ_0 is the 2×2 identity operator, $\boldsymbol{\sigma}(\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vectorial operator, $\mathbf{a}$ is generally a complex $\mathbf{C}^3$ vector and a_0 is generally a complex scalar. We shall denote the parameters $\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3$ the Stokes coefficients (parameters) of the operator A, by an extension of the well-known denomination used in polarization optics in the case of the Hermitian operators corresponding to polarizers (case in which these parameters are real).

We shall call the unit vector $\mathbf{a}/\|\mathbf{a}\|$ corresponding to $\mathbf{a}$, the *Pauli axis of the operator*. This vector plays a central role in the Pauli algebra of two-dimensional operators, and, as we shall see, in the particular cases of the unitary and Hermitian operators it reduces or it is reducible to a real $\mathbf{R}^3$ vector—the Poincaré axis of the operator—which can be visualized on the Poincaré sphere.

In the following we shall particularize the expansion (1) for some operators widespread in polarization optics: unitary operators (corresponding to the various kinds of retarders), Hermitian operators, in particular squeeze operators and projectors (corresponding to various kinds of polarizers). All these operators are normal operators (orthogonal eigenvectors) and we shall refer in this paper only to this class of operators.

2.2. Normal and nonnormal operators

One of the necessary and sufficient conditions of the normality of an operator is the commutativity with its adjoint. In our case and notations:

$$[\mathsf{A}, \mathsf{A}^\dagger] = 0, \tag{2}$$

where

$$\mathsf{A}^\dagger = \mathrm{a}_0^*\sigma_0 + \mathbf{a}^* \cdot \boldsymbol{\sigma}. \tag{3}$$

In the following we shall make largely use of the Pauli expansion of the product of two 2×2 operators [39]:

$$\mathsf{AB} = (\mathrm{a}_0 b_0 + \mathbf{a} \cdot \mathbf{b})\sigma_0 + (b_0\mathbf{a} + \mathrm{a}_0\mathbf{b}) \cdot \boldsymbol{\sigma} + \mathrm{i}(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{4}$$

This equation is an important bridge between the Hilbert space of the 2×2 linear operators and the C^3 space of their Pauli axes: by using it, the various characteristic features of these operators can be transposed in the corresponding features of their Pauli axes, reaching this way a direct geometrical interpretation.

Let us mention first a quite general result which follows straightforwardly from this relation. The commutator of two 2×2 operators A and B is determined by (the outer product of) their Pauli vectors:

$$[\mathsf{A}, \mathsf{B}] = 2\mathrm{i}(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{5}$$

Two operators commute when their Pauli vectors are collinear:

$$\mathbf{a} = \lambda\mathbf{b}, \tag{6}$$

where λ is a complex scalar.

Particularly, with equations (1), (3) and (4) in equation (2), the condition of normality of an operator reduces to

$$\mathbf{a} \times \mathbf{a}^* = 0, \tag{7}$$

i.e. the two complex-conjugate vectors $\mathbf{a}$ and $\mathbf{a}^*$ must be collinear:

$$\mathbf{a}^* = \lambda\mathbf{a}, \tag{8}$$

with λ being a complex number of modulus 1.

This condition means that, apart from a complex scalar factor, the Pauli axis of a normal operator reduces to a real vector:

$$\mathbf{a} = \mathrm{e}^{\mathrm{i}\alpha}\mathbf{r}. \tag{9}$$

Hence, with equation (1), the Pauli expansion of a *normal operator* is

$$\mathsf{N} = \mathrm{e}^{\mathrm{i}\alpha_0}\,|\mathrm{a}_0|\,\sigma_0 + \mathrm{e}^{\mathrm{i}\alpha}\mathbf{r} \cdot \boldsymbol{\sigma}, \tag{10}$$

where $\mathbf{r}$ is a R^3 vector, α_0 is a real scalar modulo 2π and α is a real scalar modulo π.

Labeling by μ the modulus of $\mathbf{r}$, and by $\mathbf{n}$ the corresponding unit vector, equation (10) may be written

$$\mathsf{N} = \mathrm{e}^{\mathrm{i}\alpha_0}\,|\mathrm{a}_0|\,\sigma_0 + \mathrm{e}^{\mathrm{i}\alpha}\mu\,\mathbf{n} \cdot \boldsymbol{\sigma}. \tag{11}$$

The Pauli axis of any normal operator is of the form:

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \mathrm{e}^{\mathrm{i}\alpha}\mathbf{n}, \tag{12}$$

where $\mathbf{n}$ is a real unit vector.

Hence the Pauli axis of a normal operator is reducible to a real unit vector. In other words, by a suitable processing, the Pauli axis of a normal operator may be brought into the real subspace R^3 of the C^3.

The Pauli axes of the nonnormal operators are irreducible complex vectors.

2.3. Unitary operators

If $\mathsf{N} = \mathsf{U}$ is a unitary operator:

$$\mathsf{U}\mathsf{U}^{\dagger} = \mathsf{I}, \tag{13}$$

(where $\mathsf{I} \equiv \sigma_0$), with equation (11) one obtains

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}\alpha_0}|a_0|\sigma_0 + \mathrm{e}^{\mathrm{i}\alpha}\mu\,\mathbf{n}\cdot\boldsymbol{\sigma})(\mathrm{e}^{-\mathrm{i}\alpha_0}|a_0|\sigma_0 + \mathrm{e}^{-\mathrm{i}\alpha}\mu\,\mathbf{n}\cdot\boldsymbol{\sigma})\\ = (|a_0|^2 + \mu^2)\sigma_0 + 2|a_0|\mu\,\mathbf{n}\cdot\boldsymbol{\sigma}\cos(\alpha - \alpha_0) = \sigma_0,\end{aligned} \tag{14}$$

wherefrom

$$|a_0|^2 + \mu^2 = 1, \tag{15}$$

$$2|a_0|\mu\,\cos(\alpha - \alpha_0) = 0. \tag{16}$$

From equation (16) we get

$$\alpha - \alpha_0 = \frac{\pi}{2}\ \text{modulo}\ \pi, \tag{17}$$

hence

$$\mathrm{e}^{\mathrm{i}(\alpha-\alpha_0)} = \pm\mathrm{i}\,. \tag{18}$$

On the other hand, equation (15) may be fulfilled if we put:

$$|a_0| = \cos\frac{\delta}{2}, \qquad \mu = \sin\frac{\delta}{2}. \tag{19}$$

Coming back to equation (11), with equations (18) and (19), the Pauli algebraic expansion of the most general unitary operator may be written in the form:

$$\mathsf{U} = \mathrm{e}^{\mathrm{i}\alpha_0}\left(\sigma_0\cos\frac{\delta}{2} \pm \mathrm{i}\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}\right) = \mathrm{e}^{\mathrm{i}\alpha_0}\,\mathrm{e}^{\pm\mathrm{i}\frac{\delta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{20}$$

In polarization optics the unitary operators correspond, as device operators, to the phase shifters (linear, circular, generally elliptical retarders).

The Pauli axis of a unitary operator:

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \pm\mathrm{i}\,\mathbf{n} \tag{21}$$

is a pure imaginary vector i.e., it is situated in the $\mathbf{I}^3$ subspace of the $\mathbf{C}^3$.

On the other hand $\mathbf{I}^3$ is isomorphic with $\mathbf{R}^3$, so that, making abstraction of the factor $\pm$i, the Pauli axis of a unitary operator may be looked as a real unit vector, as it is usually treated when is put in correspondence with the Poincaré axis of the device (e.g. a retarder), in the Poincaré sphere representation of the interaction states—polarization devices.

2.4. Hermitian operators

If $\mathsf{N} = \mathsf{H}$ is a Hermitian operator:

$$\mathsf{H} = \mathsf{H}^{\dagger}, \tag{22}$$

with equation (11) one obtains:

$$\mathrm{e}^{\mathrm{i}\alpha_0}\,|a_0|\,\sigma_0 + \mathrm{e}^{\mathrm{i}\alpha}\mu\,\mathbf{n}\cdot\boldsymbol{\sigma} = \mathrm{e}^{-\mathrm{i}\alpha_0}\,|a_0|\,\sigma_0 + \mathrm{e}^{-\mathrm{i}\alpha}\mu\,\mathbf{n}\cdot\boldsymbol{\sigma}, \tag{23}$$

wherefrom

$$\alpha_0 = 0\ \text{modulo}\ 2\pi\,; \qquad \alpha = 0\ \text{modulo}\ \pi, \tag{24}$$

so that the general Pauli algebraic form of a Hermitian operator is

$$\mathsf{H} = |a_0|\sigma_0 \pm \mu\, \mathbf{n} \cdot \boldsymbol{\sigma}, \tag{25}$$

i.e.—a well-known fact—all the Stokes coefficients of a Hermitian operator are real.

This operator may be put in an exponential form closely analogous to that of the unitary operator, equation (20). That form was obtained straightforwardly because the unitary operators are automatically unimodular; equation (15) is, in fact, a condition of unimodularity. This observation suggests the way we can adopt here.

The general Hermitian operator, equation (25), may be written as the product of the square root of its determinant with the corresponding unimodular Hermitian operator. Hence we can reduce the problem of finding the Pauli exponential expression of a general Hermitian operator to that of a Hermitian unimodular operator. The only difference between them consists in a scalar factor.

For a unimodular Hermitian operator, with equation (25), we get

$$\operatorname{Det} \mathsf{H} = |a_0|^2 - \mu^2 = 1, \tag{26}$$

equation which can be fulfilled by putting:

$$|a_0| = \cosh\frac{\eta}{2}, \qquad \mu = \sinh\frac{\eta}{2}. \tag{27}$$

Coming back to equation (25) with equation (27), the Pauli expansion of a unimodular Hermitian operator may be put in the form:

$$\mathsf{H} = \sigma_0 \cosh\frac{\eta}{2} \pm \mathbf{n} \cdot \boldsymbol{\sigma} \sinh\frac{\eta}{2} = \mathrm{e}^{\pm\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{28}$$

The unimodular Hermitian operators are largely used—under the name of *squeeze or boost operators*—in the group theoretical approach to the problems of two-state systems (including polarization), especially in the quasirelativistic approach to these problems [5, 6, 22, 23]. They pertain to the group SL(2, C), which is locally isomorphic to the $O(3, 1)$ Lorentz group.

The corresponding expansion of a general Hermitian operator may be written in a form similar to equation (20) as

$$\mathsf{H} = \mathrm{e}^{\rho}\left(\sigma_0 \cosh\frac{\eta}{2} \pm \mathbf{n} \cdot \boldsymbol{\sigma} \sinh\frac{\eta}{2}\right) = \mathrm{e}^{\rho}\mathrm{e}^{\pm\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{29}$$

As system operators, in polarization optics, the Hermitian operators describe homogeneous dichroic systems (media and polarizers).

The Pauli axis of a Hermitian operator is a real vector, it is situated in the $\mathbf{R}^3$ subspace of the $\mathbf{C}^3$. Obviously, it corresponds to the Poincaré axis of the system (e.g. partial polarizer) described by the operator, a notion which is well known in the particular case of Hermitian operators constituted by the projectors (corresponding to ideal polarizers in polarization optics).

It is gratifying to note an interesting complementarity between the Hermitian and unitary operators: their Pauli axes are situated in the complementary subspaces, $\mathbf{R}^3$ and $\mathbf{I}^3$ of the complex space $\mathbf{C}^3$. From the group theoretical viewpoint this is connected with the fact that in the case of the two-by-two representation of the Lorentz group (in which we work)—and unlike the case of the four-by-four representation—the boost generators are simply i (imaginary unit) times the rotation generators.

An important special Hermitian operator, corresponding in polarization optics to the ideal polarizers, is the orthogonal projector. It is a singular operator.

For establishing its Pauli algebraic expansion we can start with the general form of a Hermitian operator, equation (25).

The idempotency, characteristic for an orthogonal projector,

$$\mathsf{P}^2 = \mathsf{P}, \tag{30}$$

implies with equation (25):

$$(|a_0|^2 + \mu^2)\sigma_0 \pm 2|a_0|\mu\mathbf{n}\cdot\boldsymbol{\sigma} = |a_0|\sigma_0 \pm \mu\mathbf{n}\cdot\boldsymbol{\sigma}, \tag{31}$$

wherefrom

$$|a_0| = \tfrac{1}{2}, \qquad \mu = \pm\tfrac{1}{2}, \tag{32}$$

and further

$$\mathsf{P} = \tfrac{1}{2}(\sigma_0 \pm \mathbf{n}\cdot\boldsymbol{\sigma}). \tag{33}$$

Now, concerning the density operators of the polarization states of the light, the polarization density operator of a pure state (totally polarized light) of unit intensity has a form [35] similar to equation (33)—such a state can be produced by an ideal polarizer (33):

$$\mathsf{J} = \tfrac{1}{2}(\sigma_0 + \mathbf{s}\cdot\boldsymbol{\sigma}), \tag{34}$$

where $\mathbf{s}$ is the Poincaré axis of the state.

The Pauli algebraic form of the polarization density operator of a mixed state (partially polarized light) is [43]

$$\mathsf{J} = \frac{\mathscr{I}}{2}(\sigma_0 + p\mathbf{s}\cdot\boldsymbol{\sigma}). \tag{35}$$

The three parameters which characterize the beam of light—the intensity $\mathscr{I}$, the degree of polarization, p, and the Poincaré unit vector of its polarization state $\mathbf{s}$ (or, globally, the Poincaré vector $p\mathbf{s}$ of its state)—appear all, in block, in the expression of the density operator of the state. Generally all these parameters are modified by the device.

3. Interaction of light with the canonical polarization devices

The action of a polarization system (device/medium) on the polarized light may be analyzed at the level of the light spinors only for the pure states (totally polarized light). For mixed states (partially polarized light) it can be analyzed only at the level of the density operator of the state, where it takes the operatorial form [3, 4]:

$$\mathsf{J}_o = \mathsf{D}\mathsf{J}_i\mathsf{D}^\dagger, \tag{36}$$

irrespective of the algebra in which we handle this action. Here D is the operator of the device and J_i and J_o the polarization density operators of the incident and emergent light respectively.

If we consider the polarization density operator of the incident light normalized to unit intensity:

$$\mathsf{J}_i = \tfrac{1}{2}(\sigma_0 + p_i\,\mathbf{s}_i\cdot\boldsymbol{\sigma}), \tag{37}$$

then the density operator of the emergent (output) light has the generic form:

$$\mathsf{J}_o = \tfrac{1}{2}g(\sigma_0 + p_o\,\mathbf{s}_o\cdot\boldsymbol{\sigma}). \tag{38}$$

Here g is the ratio of the intensity of the emergent light to that of the incident light, the so-called gain of the transformation [3, 4]; it is subunitary or overunitary, according to whether the medium is absorbent or amplifier. We have to point out that our approach, unlike the standard ones, uses the primary definition of the gain.

The algorithm of this vectorial Pauli algebraic approach we use for analyzing the interaction of the light with the optical devices/media is the following:

- one gives the characteristics of the incident light, p_i, $\mathbf{s}_i$ embodied in its polarization density operator, equation (37),
- one gives the characteristics of the device—e.g. α_0, δ, $\mathbf{n}$, equation (20)—embodied in its operator,
- we apply the general law of transformation of the polarization density operator, equation (36), and we get the characteristics of the emergent light p_o, $\mathbf{s}_o$ and g, embodied in its polarization density operator. All the characteristics of the emergent light result this way straightforwardly, compactly, in block.

We shall analyze here, the interactions of the homogeneous birefringent and dichroic devices ('canonical' devices [4]) with the partially polarized light.

3.1. Action of a phase shifter on the partially polarized light

Under the action of the operator, equation (20), corresponding to the phase shifter, the density operator of a mixed state (partially polarized light) of Poincaré axis $\mathbf{s}_i$, equation (37), becomes

$$\mathsf{J}_o = \mathsf{U}_{\mathbf{n}}(\delta)\mathsf{J}_i\mathsf{U}_{\mathbf{n}}^{\dagger}(\delta). \tag{39}$$

It is worth noting that the phase factor $\mathrm{e}^{\mathrm{i}\alpha_o}$ which appears in the general form of the unitary operator plays no role in this action; it is eliminated in equation (39).

$$\begin{aligned}\mathsf{J}_o &= \frac{1}{2}\left(\sigma_0\cos\frac{\delta}{2} - \mathrm{i}\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}\right)(\sigma_0 + p_i\mathbf{s}_i\cdot\boldsymbol{\sigma})\left(\sigma_0\cos\frac{\delta}{2} + \mathrm{i}\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}\right)\\ &= \frac{1}{2}\Bigg[\sigma_0 + p_i\mathbf{s}_i\cdot\boldsymbol{\sigma}\left(\cos^2\frac{\delta}{2} - \sin^2\frac{\delta}{2}\right)\\ &\quad + 2p_i(\mathbf{n}\times\mathbf{s}_i)\cdot\boldsymbol{\sigma}\sin\frac{\delta}{2}\cos\frac{\delta}{2} + 2p_i\mathbf{n}\cdot\mathbf{s}_i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\delta}{2}\Bigg]\\ &= \frac{1}{2}\left\{\sigma_0 + p_i\left[\mathbf{s}_i\cdot\boldsymbol{\sigma}\cos\delta + (\mathbf{n}\times\mathbf{s}_i)\cdot\boldsymbol{\sigma}\sin\delta + 2\mathbf{n}\cdot\mathbf{s}_i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin^2\frac{\delta}{2}\right]\right\}.\end{aligned} \tag{40}$$

We have used here Dirac's identity:

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b} + \mathrm{i}(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}, \tag{41}$$

where $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional vectors.

The density operator of the output light has a form corresponding to a mixed state:

$$\mathsf{J}_o = \tfrac{1}{2}(\sigma_0 + p_o\mathbf{s}_o\cdot\boldsymbol{\sigma}), \tag{42}$$

where

$$g = 1, \tag{43}$$

$$p_o = p_i, \tag{44}$$

and

$$\mathbf{s}_o = \mathbf{s}_i\cos\delta + \mathbf{n}\times\mathbf{s}_i\sin\delta + (\mathbf{n}\cdot\mathbf{s}_i)\mathbf{n}(1-\cos\delta). \tag{45}$$

Let us emphasize the compactness of this approach: all the characteristics of the emergent light, $\mathbf{s}_o$, p_o, g, are contained in the single (vectorial) result, equation (40).

Equations (43) and (44) show that any (generally elliptic) retarder changes the state of polarization of the incident light without affecting its intensity and degree of polarization. This is a well-known experimental fact, confirmed theoretically by various approaches to the interaction polarization states–polarization systems [1–4].

Equation (45) constitutes a new result of our approach. It expresses the Poincaré unit vector of the emergent light, $\mathbf{s}_o$, as a vectorial function of the Poincaré unit vector of the incident light, $\mathbf{s}_i$, and the parameters of the birefringent system (its Poincaré axis $\mathbf{n}$ and its retardance δ).

Let us get a more intuitive grasp of this equation. Equation (45) gives the law of transformation of the Poincaré axis of a mixed state under the action of a polarization device described by a unitary operator on this state. It expresses a rotation in $\mathbf{R}^3$, here a rotation of angle δ of the unit vector $\mathbf{s}_i$ around the Poincaré axis $\mathbf{n}$ of the device.

Indeed, we can recognize in equation (45) the operator of the $\mathbf{R}^3$ rotation [44]:

$$\mathbf{s}_0 = \mathsf{R}_{\mathbf{n}}(\delta)\mathbf{s}_i. \tag{46}$$

$$\mathsf{R}_{\mathbf{n}}(\delta) = \cos\delta + (1-\cos\delta)\mathbf{n}(\mathbf{n}\cdot) + \sin\delta(\mathbf{n}\times). \tag{47}$$

Multiplying equation (45) at left with p_o and at right with p_i—on the basis of the equality (44), it follows that the Poincaré vector of the incident light, $p_i\ \mathbf{s}_i$, is rotated by the retarder on the Poincaré inner sphere $\sum_2^{p_i}$.

If the incident light is totally polarized, $p_i = 1$, then conformly to (44) $p_o = 1$ and the Poincaré vector of the incident light is rotated on the sphere $\sum_2^1$.

It is gratifying to emphasize that the unitary operator $\mathsf{U}_{\mathbf{n}}$, equation (20), acts, equation (39), on the state of optical polarization expressed by its polarization density operator, hence in the Hilbert space of the polarization density operators of the states, whereas the rotation operator $\mathsf{R}_{\mathbf{n}}$, equation (47), acts on a $\mathbf{R}^3$ vector, namely the Poincaré vector, $\mathbf{s}_i$, abstractly associated with this state.

The main result of this section is establishing an algebraic bridge between the Hilbert space of the (density operators of the) states of optical polarization and the Σ_3^1 space of their Poincaré geometric ($\mathbf{R}^3$) representation: each unitary action, equation (20), in the first space is mapped in a rotation, equation (47), in the second space. This is the mathematical ground when handling the actions of the phase shifters as rotations on the Poincaré sphere. In fact, obviously, a phase shifter acts, in the real physical space, on the electric vector of the light wave. This action is mapped in a unitary action $\mathsf{U}_{\mathbf{n}}(\delta)$ in the Hilbert space of the density operators of the states and further in a rotation $\mathsf{R}_{\mathbf{n}}(\delta)$ on the Poincaré sphere Σ_3^1 (more generally $\Sigma_3^{p_i}$).

3.2. Action of an orthogonal dichroic device on the partially polarized light

Referring to the most general Pauli algebraic form of a Hermitian operator (29), in polarization optics the quantities:

$$\mathrm{e}^{\rho} = \mathrm{e}^{\frac{\eta_1+\eta_2}{2}}, \tag{48}$$

$$\mathrm{e}^{\eta} = \mathrm{e}^{\eta_1-\eta_2} \tag{49}$$

are the isotropic and the relative amplitude transmittances of the dichroic device, respectively, where e^{η_1} and e^{η_2} are its principal (eigen-) transmittances. The coefficients η_1 and η_2 may be, each of them, positive as well as negative. For fixing the ideas, in the case of diattenuators both are negative and consequently the overall transmittance is subunitary.

We shall take the Poincaré axis $\mathbf{n}$ of the dichroic device as corresponding to its major eigenstate (let us define it as the state of maximum transmittance irrespective of the fact that the device is a diamplificator or a diattenuator).

With

$$\mathsf{J}_o = \mathsf{H}_{\mathbf{n}} \mathsf{J}_i \mathsf{H}_{\mathbf{n}}^{\dagger}, \tag{50}$$

after some algebra similar to equation (40) and using Dirac's identity, we get

$$\begin{aligned} \mathsf{J}_o &= \frac{1}{2} \mathrm{e}^{2\rho} \left(\sigma_0 \cosh \frac{\eta}{2} + \mathbf{n} \cdot \boldsymbol{\sigma} \sinh \frac{\eta}{2} \right) (\sigma_0 + p_i \mathbf{s}_i \cdot \boldsymbol{\sigma}) \left(\sigma_0 \cosh \frac{\eta}{2} + \mathbf{n} \cdot \boldsymbol{\sigma} \sinh \frac{\eta}{2} \right) \\ &= \frac{1}{2} \mathrm{e}^{2\rho} \left\{ \sigma_0 (\cosh \eta + p_i \mathbf{s}_i \cdot \mathbf{n} \sinh \eta) + \left[p_i \mathbf{s}_i + \mathbf{n} \sinh \eta + 2 p_i (\mathbf{n} \cdot \mathbf{s}_i) \mathbf{n} \sinh^2 \frac{\eta}{2} \right] \cdot \boldsymbol{\sigma} \right\}. \end{aligned} \tag{51}$$

This expression is that of a mixed state:

$$\mathsf{J}_o = \tfrac{1}{2} g [\sigma_0 + p_o \mathbf{s}_o \cdot \boldsymbol{\sigma}], \tag{52}$$

where g is the gain given by the dichroic device:

$$g = \mathrm{e}^{2\rho} (\cosh \eta + p_i \mathbf{s}_i \cdot \mathbf{n} \sinh \eta) = \mathrm{e}^{2\eta_1} \frac{1 + p_i \cos \alpha}{2} + \mathrm{e}^{2\eta_2} \frac{1 - p_i \cos \alpha}{2} \tag{53}$$

and the degree of polarization and the Poincaré unit vector of the output light are given by

$$p_o \mathbf{s}_o = \frac{p_i \mathbf{s}_i + \mathbf{n} \sinh \eta + 2 p_i (\mathbf{n} \cdot \mathbf{s}_i) \mathbf{n} \sinh^2 \frac{\eta}{2}}{\cosh \eta + p_i \mathbf{s}_i \cdot \mathbf{n} \sinh \eta}. \tag{54}$$

We have labeled in equation (53) by α the angle between the Poincaré unit vectors of the incident light and of the device, $\mathbf{s}_i$ and $\mathbf{n}$.

Again all the information about the action of the dichroic device on the partially polarized incident light is given very compactly, in block, by equation (51), or equivalently by equations (53) and (54).

Let us consider now some particular cases of the results (51)–(54).

If the state of polarization of the incident light coincide with the major eigenstate of the dichroic device, the Poincaré vectors of the device and of the state are parallel:

$$\mathbf{s}_i = \mathbf{n}, \tag{55}$$

case in which:

$$\begin{aligned} \mathsf{J}_o &= \mathrm{e}^{2\rho} \frac{1}{2} \left[\sigma_0 (\cosh \eta + p_i \sinh \eta) + \left(p_i + \sinh \eta + 2 p_i \sinh^2 \frac{\eta}{2} \right) \mathbf{n} \cdot \boldsymbol{\sigma} \right] \\ &= \mathrm{e}^{2\rho} \frac{1}{2} (\cosh \eta + p_i \sinh \eta) \left[\sigma_0 + \frac{\tanh \eta + p_i}{1 + p_i \tanh \eta} \mathbf{n} \cdot \boldsymbol{\sigma} \right]. \end{aligned} \tag{56}$$

In this particular case it is straightforward to separate p_o and $\mathbf{s}_o$. In equation (56) $\mathbf{n}$ is a unit vector, so that:

$$\mathbf{s}_o = \mathbf{n} = \mathbf{s}_i. \tag{57}$$

The emergent light is partially polarized in the major eigenstate of the device (coincident with the state of the incident light) and its intensity is reduced or amplified by

$$g_{\max} = \mathrm{e}^{2\rho} (\cosh \eta + p_i \sinh \eta) = \mathrm{e}^{2\eta_1} \frac{1 + p_i}{2} + \mathrm{e}^{2\eta_2} \frac{1 - p_i}{2}. \tag{58}$$

The degree of polarization of the output state depends both of the degree of polarization of the input state, p_i, and of the coefficient of relative transmittance of the device, η:

$$p_o = \frac{\sinh \eta + p_i \cosh \eta}{\cosh \eta + p_i \sinh \eta} = \frac{\tanh \eta + p_i}{1 + p_i \tanh \eta}. \tag{59}$$

If, in contrast, the polarization state of the incident light is 'aligned' with the minor eigenstate of the device:

$$\mathbf{s}_i = -\mathbf{n}, \tag{60}$$

then equations (51), (53) and (54) give

$$\mathsf{J}_o = \mathrm{e}^{2\rho}\frac{1}{2}(\cosh\eta - p_i\sinh\eta)\left[\sigma_0 + \frac{\tanh\eta - p_i}{1 - p_i\tanh\eta}\mathbf{n}\cdot\boldsymbol{\sigma}\right]. \tag{61}$$

The degree of polarization, the modulus of p_o, and the gain are given by

$$p_o = \frac{\sinh\eta - p_i\cosh\eta}{\cosh\eta - p_i\sinh\eta} = \frac{\tanh\eta - p_i}{1 - p_i\tanh\eta}, \tag{62}$$

$$g_{\min} = \mathrm{e}^{2\rho}(\cosh\eta - p_i\sinh\eta) = \mathrm{e}^{2\eta_1}\frac{1 - p_i}{2} + \mathrm{e}^{2\eta_2}\frac{1 + p_i}{2}. \tag{63}$$

It is interesting to note that formulae (59) and (62) are strongly similar with some formulae in the theory of special relativity [45]. This similarity is not a casual one. Its roots stand in the isomorphism between the group of transformations SL(2, C) and the Lorentz group $O(3, 1)$ which describe the transformations of the special relativity. Well-known, this isomorphism was largely exploited in the last decade in the quasirelativistic formulation of the theory of polarization and, more generally, of the 'two-state' ('two-beam') systems [5, 6, 21–23].

Let us come back to the general results, equations (53) and (54), of our approach. All information concerning the interaction of totally polarized, unpolarized or partially polarized light with the ideal or partial polarizers is contained in these two compact formulae.

The expression (53) of the gain constitutes *the largest generalization of Malus' law*, valid for partially polarized light passed through any canonical dichroic device. To our best knowledge it is a new result, obtained by our approach.

In the general case, the separation of the degree of polarization and of the Poincaré unit vector of the output light—in equation (54)—is not so straightforward. The final expressions are

$$p_o = \left[1 - \frac{1 - p_i^2}{(\cosh\eta + p_i\mathbf{n}\cdot\mathbf{s}_i\sinh\eta)^2}\right]^{1/2}, \tag{64}$$

$$\mathbf{s}_o = \frac{p_i\mathbf{s}_i + \mathbf{n}\sinh\eta + 2p_i(\mathbf{n}\cdot\mathbf{s}_i)\mathbf{n}\sinh^2\frac{\eta}{2}}{\left[p_i^2 - 1 + (\cosh\eta + p_i\mathbf{n}\cdot\mathbf{s}_i\sinh\eta)^2\right]^{1/2}}. \tag{65}$$

Instead of presenting here the corresponding calculus we shall point out that, in fact, the vector $p_o\mathbf{s}_o$ gives a complete characterization of the arbitrarily polarized state of light. We are not interested separately in p_o and $\mathbf{s}_o$. The state vector $p_o\mathbf{s}_o$ points in the Poincaré direction of the (totally polarized component of the) mixed state, and its top is situated on the $\Sigma_2^{p_o}$ sphere corresponding to the degree of polarization p_o of the state.

It is gratifying to note that in the limit $\eta_1 \to 0$, $\eta_2 \to -\infty$, $\eta \to \infty$, when the diattenuator passes in an ideal polarizer (pure projector), equations (51), (53) and (54) give the well-known results corresponding to the action of an orthogonal projector (ideal polarizer) on a mixed state, particularly some reduced forms of Malus' law.

4. Conclusions

The Pauli algebra has been used for a long time in polarization optics, mainly in its scalar and usually in its matrix form, for describing the states of polarization of the light and its interaction with various polarization systems (devices, media).

In this paper we have used the Pauli algebra in its very compact, vectorial and pure operatorial form, for analyzing the action of the canonical devices on the partially polarized light.

A first, visible, advantage of the vectorial Pauli algebraic approach in comparison with the traditional methods arises from its compactness. Unlike in the standard approaches, the characteristics of the emergent light—its Poincaré unit vector $\mathbf{s}_o$, the polarization degree p_o and the intensity—result straightforwardly, in block, e.g. equations (40) and (51).

A second, deeper, advantage of the vectorial Pauli algebraic approach is that it is intimately connected with the Poincaré sphere representation, which plays a central role in the theory and practice of polarization. The vectorial Pauli algebraic approach brings into focus the Pauli axes of the operators, which have a direct correspondent in the Poincaré ball geometric representation: the Poincaré vectors of the polarization states and of the devices. Consequently, the vectorial Pauli algebra provides the shortest bridge between the actions of the device operators in the Hilbert space of the (polarization density operators of the) states and the corresponding movements of the Poincaré vectors associated with these states on the Poincaré sphere or in the Poincaré ball. We have illustrated this fact firstly in section 3.1, by establishing the rotation operator $\mathsf{R}_{\mathbf{n}}$ in the $\mathbf{R}^3$ Poincaré space corresponding to the unitary operator $\mathsf{U}_{\mathbf{n}}$ of a phase shifter, which acts in the Hilbert space of the polarization states. This connection is also straightforward in the second case we have analyzed, that of the Hermitian operators corresponding to the dichroic devices.

Another advantage of the vectorial Pauli algebraic method, which became visible in the analysis of the action of the dichroic devices on partially polarized light (section 3.2) is that the results have a high degree of symmetry—e.g. equations (53), (58) and (63). This is because our approach is parameterized in a manner well adapted to both the symmetries of the polarization states space and of the devices: On the one hand, this approach is performed in the Hilbert space of the (density operators of the) polarization states and in the Poincaré $\sum_3^1$ space, isomorphic with the first. Well known, the description of a physical system in the abstract space of the states of the system (e.g. the space of configurations) reflects better than its description in the real physical space its essential properties and symmetries. This is the case of the Hilbert space of the states too. On the other hand, concerning the devices, our approach addresses their eigenstates—the Poincaré axis of the device operator joints its eigenstates—and their eigenvalues—e.g. equations (48) and (49)—which both reflect the symmetry of the device. Particularly, the generalized Malus' law (for any dichroic device and any state of polarization) we have established, equation (53), also has a very symmetric form.

Finally, we have to point out that the field of applicability of the formalism we have given in this paper is much larger than that of light polarization. It can be applied, in the same form, to any 'two-state' ('two-beam') system (interferometric systems, multilayer systems, geometrical optic systems, spin $1/2$ systems, a.s.o.).

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